The University of Texas at Austin
Electrical and Computer Engineering
Cockrell School of Engineering

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INTRODUCTION TO COMPUTER VISION

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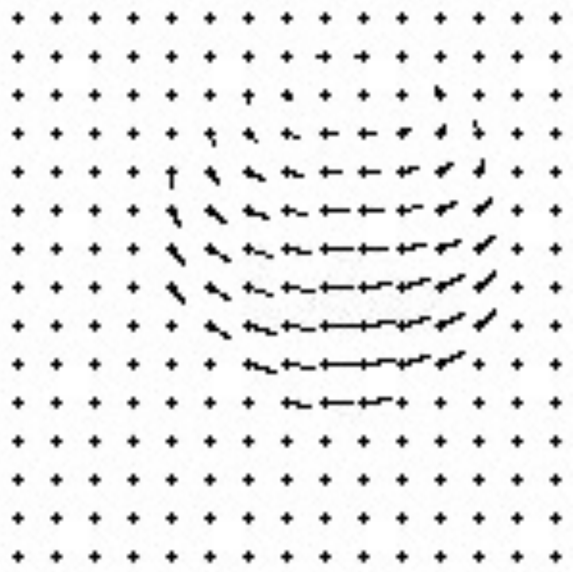
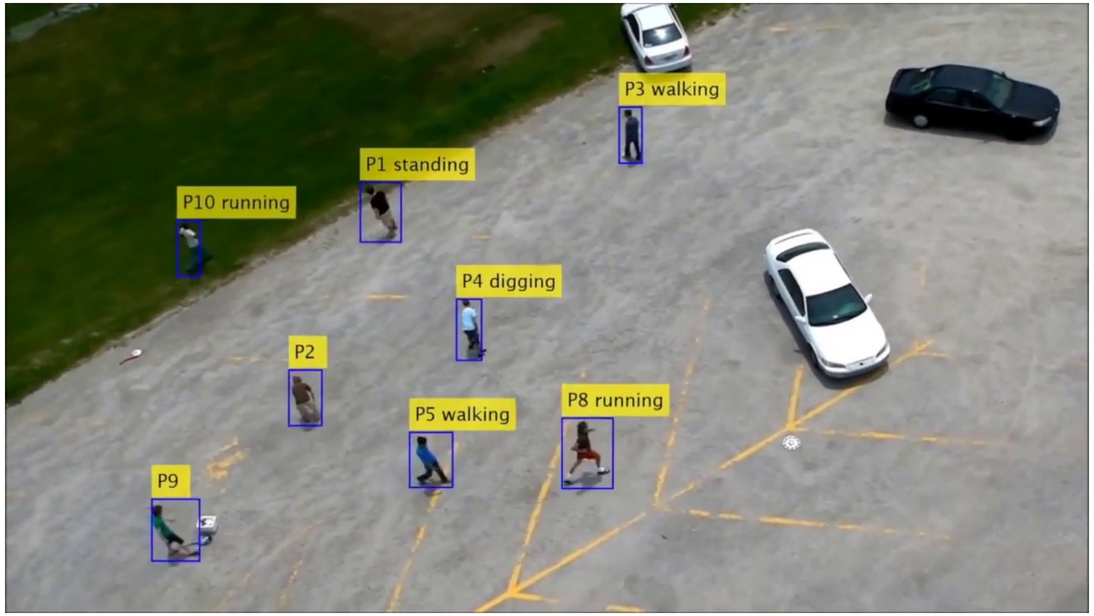
https://vita-group.github.io/

Finally: Motion and Video!

Tracking objects, video analysis, low level motion





Motion vs. Stereo: Similarities/Differences

- **Both involve solving**
  - Correspondence: disparities, motion vectors
  - Reconstruction

- **Motion:**
  - Uses velocity: consecutive frames must be close to get good approximate time derivative
  - 3d movement between camera and scene not necessarily single 3d rigid transformation

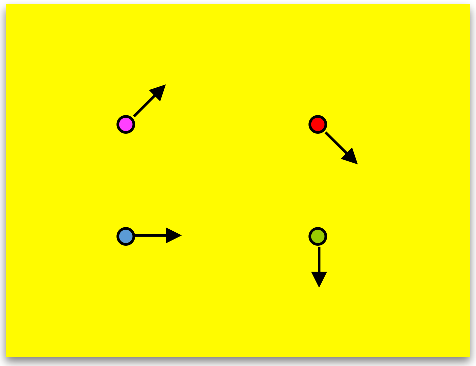
- **Whereas with stereo:**
  - Could have any disparity value
  - View pair separated by a single 3d transformation

Slide taken from Kristen Grauman

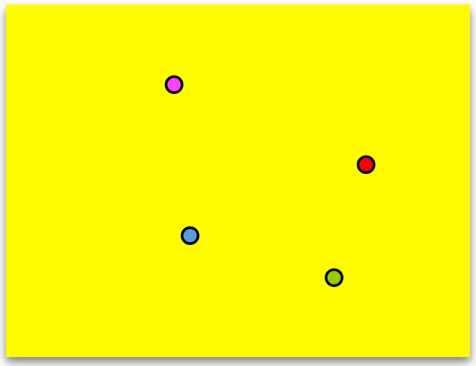
Today We Focus on: Optical Flow

Problem Definition

Given two consecutive image frames, estimate the motion of each pixel



$I(x, y, t)$



$I(x, y, t')$

Estimate the motion (flow) between these two consecutive images

Visual Example

Key Assumptions

(unique to optical flow & different from generally estimating two image view transforms!)

Color Constancy

(Brightness constancy for intensity images)

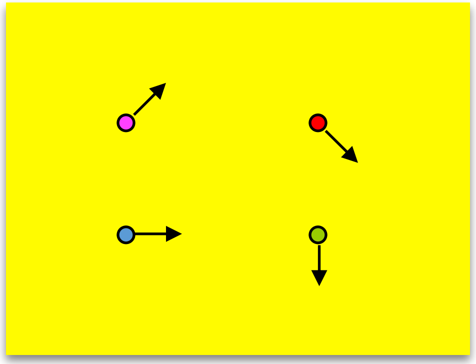
Implication: allows for pixel to pixel comparison
(not image features)

Small Motion

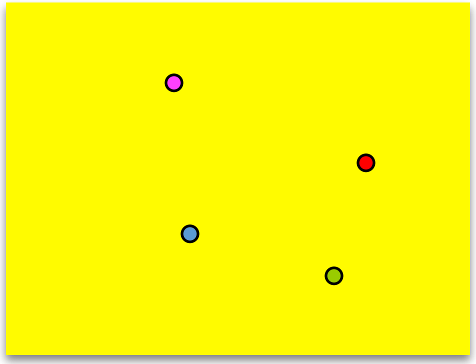
(pixels only move a little bit)

Implication: linearization of the brightness
constancy constraint

Approach



$I(x, y, t)$



$I(x, y, t')$

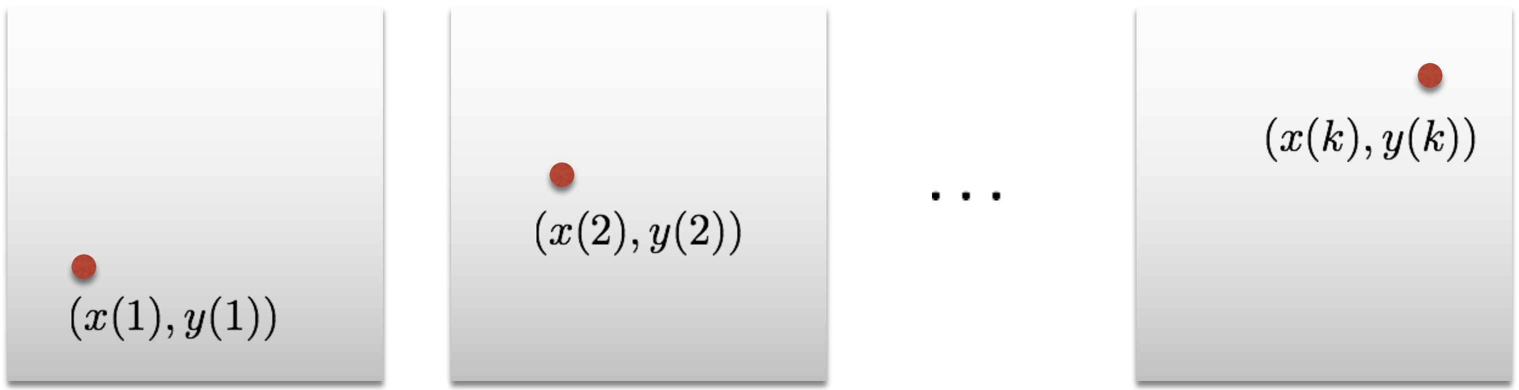
Look for nearby pixels with the same color

(small motion) (color constancy)

Assumption 1

Brightness constancy

Scene point moving through image sequence



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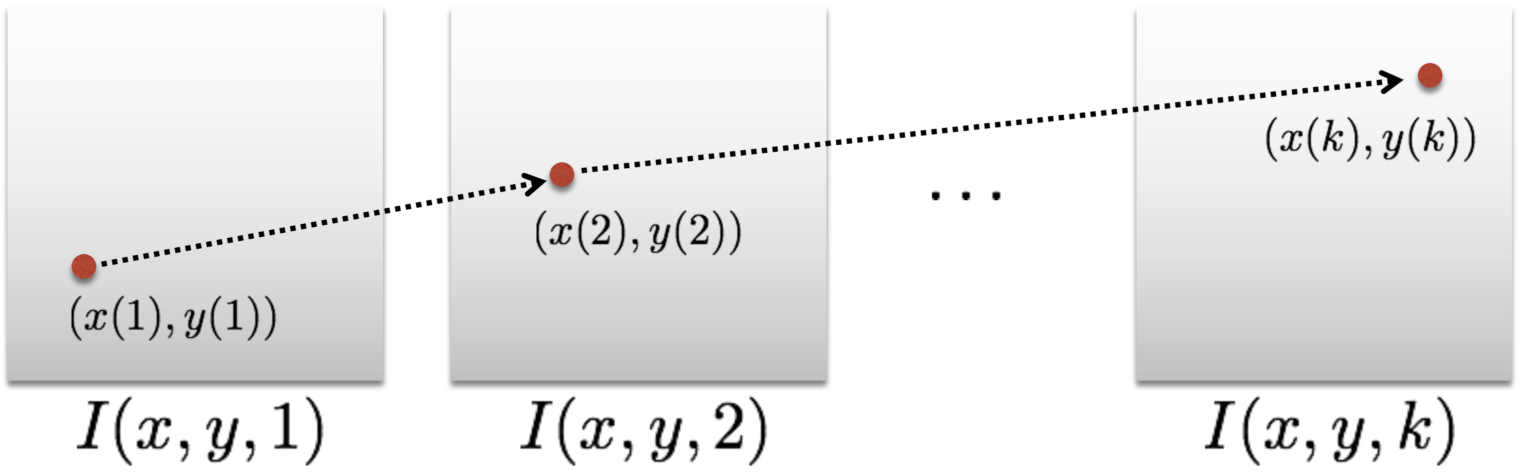

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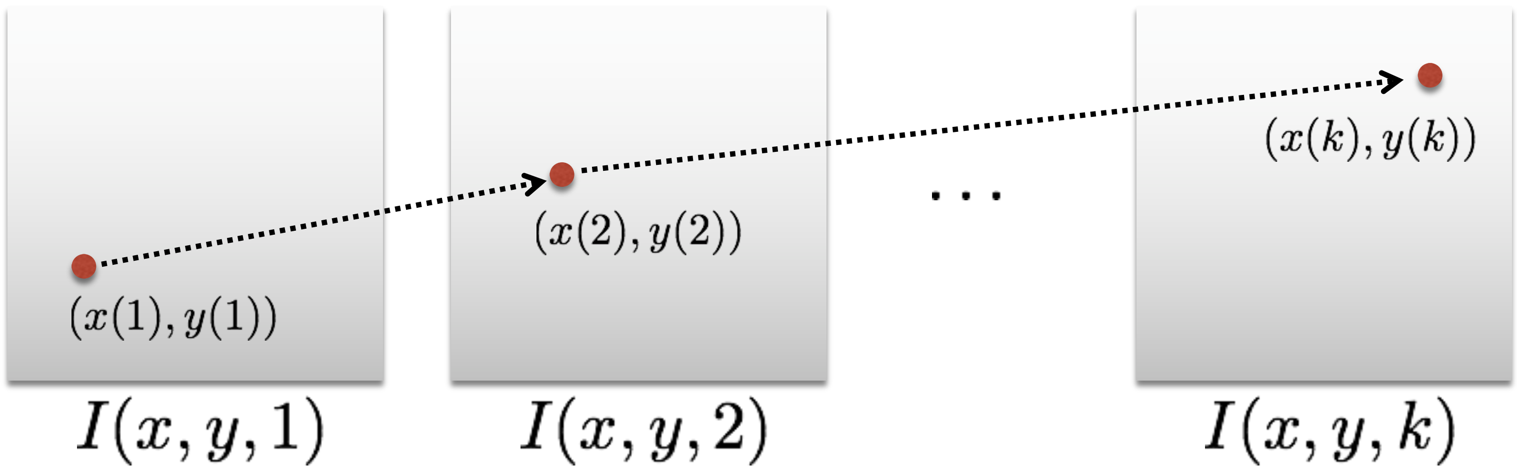
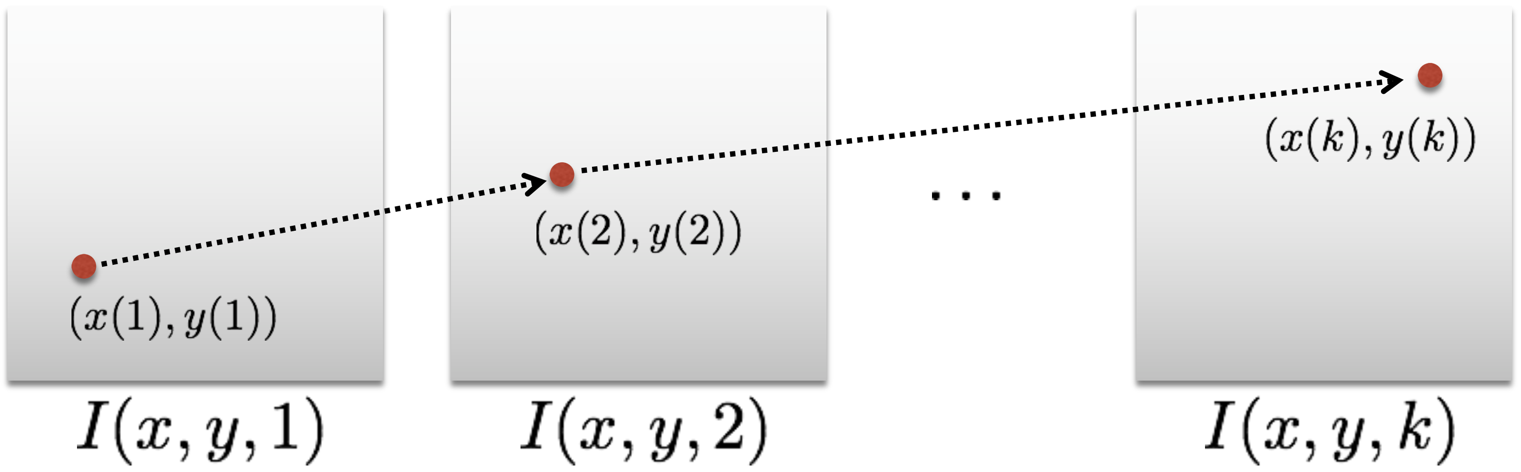


Assumption:Brightness of the point will remain the same

Assumption 1

Brightness constancy

Scene point moving through image sequence

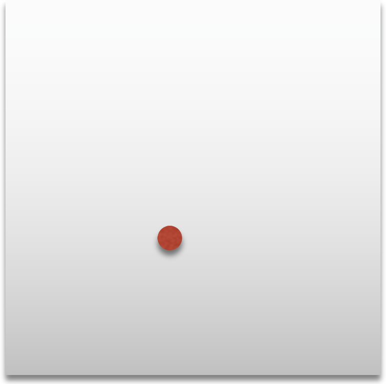


Assumption:Brightness of the point will remain the same

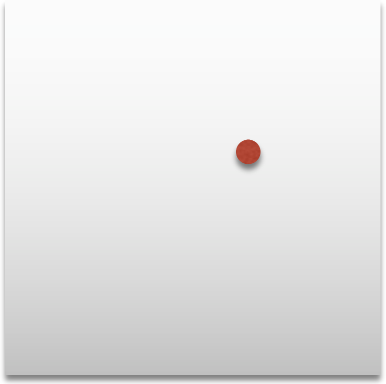
$$\underbrace{I(x(t), y(t), t) = C}_{\text{constant}}$$

Assumption 2

Small motion



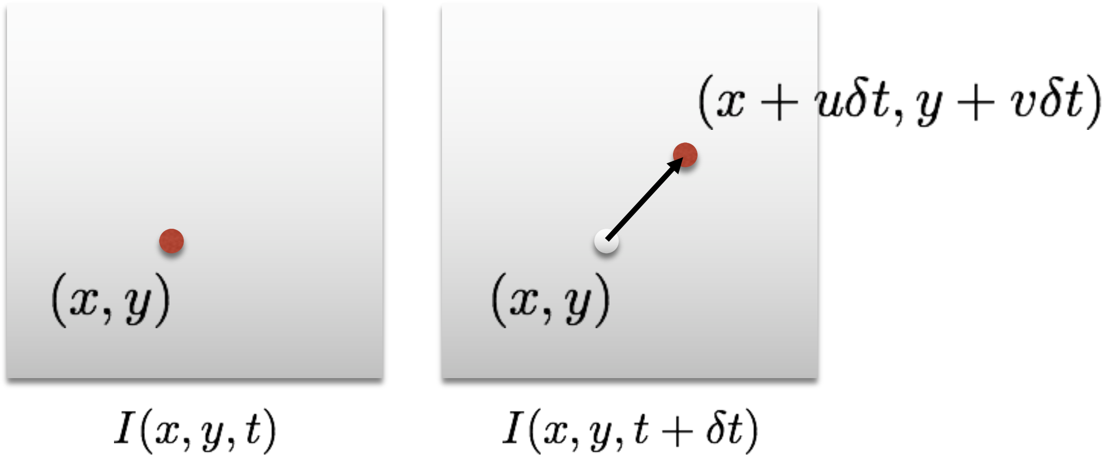
$I(x, y, t)$



$I(x, y, t + \delta t)$

Assumption 2

Small motion



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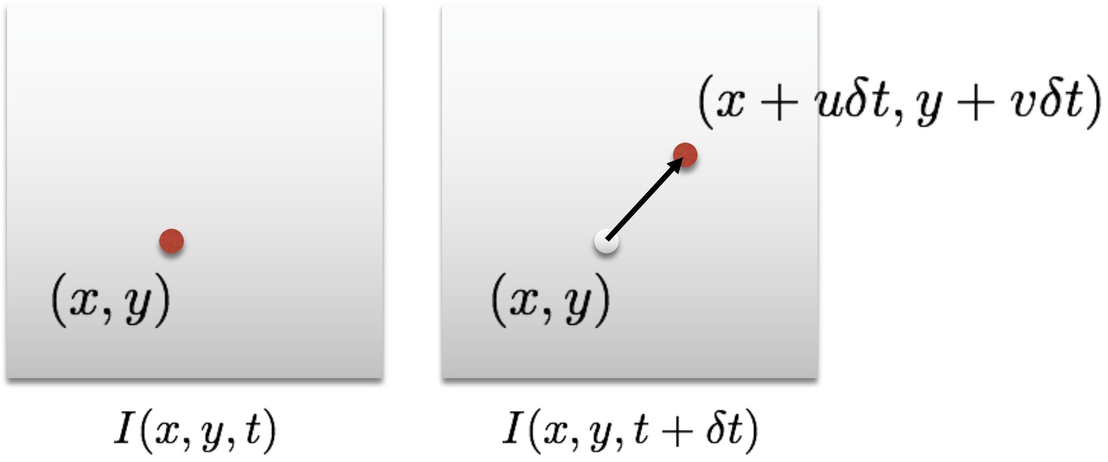
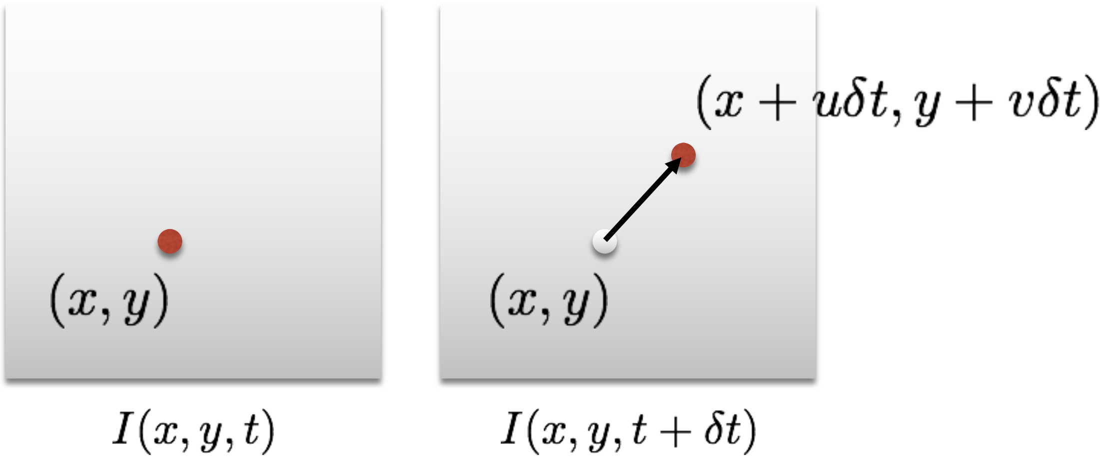


Optical flow (velocities): (u, v) Displacement: $(\delta x, \delta y) = (u\delta t, v\delta t)$

Assumption 2

Small motion



Optical flow (velocities): (u, v) Displacement: $(\delta x, \delta y) = (u\delta t, v\delta t)$

For a ***really small space-time step***…

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

… the brightness between two consecutive image frames is the same

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

Insight:

If the time step is really small,

we can *linearize* the intensity function

$$I(x+u\delta t, y+v\delta t, t+\delta t) = I(x,y,t)$$

Multivariable Taylor Series Expansion
(First order approximation, three variables)

$$I(x,y,t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x,y,t)$$

assuming small motion

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0$$

divide by δt

take limit $\delta t \rightarrow 0$

$$\boxed{\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0}$$

Brightness Constancy Equation

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

Brightness Constancy Equation

$$I_x u + I_y v + I_t = 0$$

(x-flow) (y-flow)

shorthand notation

$$\nabla I^\top \boldsymbol{v} + I_t = 0$$

(1 x 2) (2 x 1)

vector form

(putting the math aside for a second…)

What do the terms of the
brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$

(putting the math aside for a second…)

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brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$

Image gradients
(at a point p)

(putting the math aside for a second…)

What do the term of the
brightness constancy equation represent?

flow velocities

$$I_x u + I_y v + I_t = 0$$

Image gradients
(at a point p)

(putting the math aside for a second…)

What do the term of the
brightness constancy equation represent?

flow velocities

$$I_x u + I_y v + I_t = 0$$

Image gradients
(at a point p)

temporal gradient

How do you compute these terms?

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

Forward difference
Sobel filter
Derivative-of-Gaussian filter
…

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
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…

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…

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

frame differencing

Frame differencing

t

1	1	1	1	1
1	1	1	1	1
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10

-

$t+1$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	10	10	10
1	1	10	10	10
1	1	10	10	10

=

$$I_t = \frac{\partial I}{\partial t}$$

0	0	0	0	0
0	0	0	0	0
0	9	9	9	9
0	9	0	0	0
0	9	0	0	0
0	9	0	0	0

(example of a forward difference)

Example:

3 x 3 patch

t

1	1	1	1	1
1	1	1	1	1
1	**10**	**10**	**10**	**10**
1	**10**	**10**	**10**	**10**
1	**10**	**10**	**10**	**10**
1	**10**	**10**	**10**	**10**

$t + 1$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	**10**	**10**	**10**
1	1	**10**	**10**	**10**
1	1	**10**	**10**	**10**

$$I_x = \frac{\partial I}{\partial x}$$

-	0	0	0	-
-	0	0	0	-
-	**9**	0	0	-
-	**9**	0	0	-
-	**9**	0	0	-
-	**9**	0	0	-

-1 0 1

$$I_y = \frac{\partial I}{\partial y}$$

-	-	-	-	-
0	0	0	0	0
0	**9**	**9**	**9**	**9**
-	0	0	0	0
0	0	0	0	0
-	-	-	-	-

-1
0
1

$$I_t = \frac{\partial I}{\partial t}$$

0	0	0	0	0
0	0	0	0	0
0	**9**	**9**	**9**	**9**
0	**9**	0	0	0
0	**9**	0	0	0
0	**9**	0	0	0

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

Forward difference
Sobel filter
Derivative-of-Gaussian filter
…

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

optical flow

How do you compute this?

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

optical flow

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
Sobel filter
Derivative-of-Gaussian filter
…

We need to solve for this!
(this is the unknown in the optical flow problem)

frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute …

spatial derivative	optical flow	temporal derivative
$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$	$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$	$I_t = \frac{\partial I}{\partial t}$

spatial derivative

Forward difference
Sobel filter
Derivative-of-Gaussian filter
…

optical flow

(u, v)

Solution lies on a line

Cannot be found uniquely with a single constraint

temporal derivative

frame differencing

unknown

$$I_x u + I_y v + I_t = 0$$

known

We need at least ____ equations to solve for 2 unknowns.

unknown

$$I_x u + I_y v + I_t = 0$$

known

Where do we get more equations (constraints)?

Horn-Schunck
Optical Flow (1981)

brightness constancy

small motion

'smooth' flow
(flow can vary small from pixel to pixel)

global method
(dense)

Lucas-Kanade
Optical Flow (1981)

method of differences

'constant' flow
(flow is constant for all nearby pixels)

local method
(sparse)

Where do we get more equations (constraints)?

$$I_x u + I_y v + I_t = 0$$

Assume that the surrounding patch (say 5x5) has
‘**constant flow**’

Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us 25 equations

$$
\begin{gathered}
I_x(\boldsymbol{p}_1)u + I_y(\boldsymbol{p}_1)v = -I_t(\boldsymbol{p}_1) \\
I_x(\boldsymbol{p}_2)u + I_y(\boldsymbol{p}_2)v = -I_t(\boldsymbol{p}_2) \\
\vdots \\
I_x(\boldsymbol{p}_{25})u + I_y(\boldsymbol{p}_{25})v = -I_t(\boldsymbol{p}_{25})
\end{gathered}
$$

In General, How Many Solutions?

Equivalent to solving:

$$\underbrace{\begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}}_{A^\top A} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\hat{x}} = - \underbrace{\begin{bmatrix} \sum_{p \in P} I_x I_t \\ \sum_{p \in P} I_y I_t \end{bmatrix}}_{A^\top b}$$

where the summation is over each pixel $\boldsymbol{p}$ in patch $\boldsymbol{P}$

$$x = (A^\top A)^{-1} A^\top b$$

Called "Pseudo Inverse"

When is this solvable?

$$A^\top A\hat{x} = A^\top b$$

When is this solvable?

$$A^\top A\hat{x} = A^\top b$$

$A^\mathsf{T}A$ should be invertible

$A^\mathsf{T}A$ should not be too small

λ_1 and λ_2 should not be too small

$A^\mathsf{T}A$ should be well conditioned

λ_1/λ_2 should not be too large (λ_1=larger eigenvalue)

Where have you seen this before?

$$A^\top A = \begin{bmatrix} \sum\limits_{p \in P} I_x I_x & \sum\limits_{p \in P} I_x I_y \\ \sum\limits_{p \in P} I_y I_x & \sum\limits_{p \in P} I_y I_y \end{bmatrix}$$

Where have you seen this before?

$$A^\top A = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

Harris Corner Detector!

Where have you seen this before?

$$A^\top A = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

Harris Corner Detector!

What are the implications?

Implications

- Corners are when $\lambda 1$, $\lambda 2$ are big; this is also when Lucas-Kanade optical flow works best
- Corners are regions with two different directions of gradient (at least)
- **Corners are good places to compute flow!**
- That is why Lucas-Kanade flow is considered “local/sparse”

What happens when you have no ‘corners’?

You want to compute optical flow.
What happens if the image patch contains only a line?

Horn-Schunck Optical Flow (1981)

brightness constancy

small motion

'smooth' flow

(flow can vary from pixel to pixel)

global method
(dense)

Lucas-Kanade Optical Flow (1981)

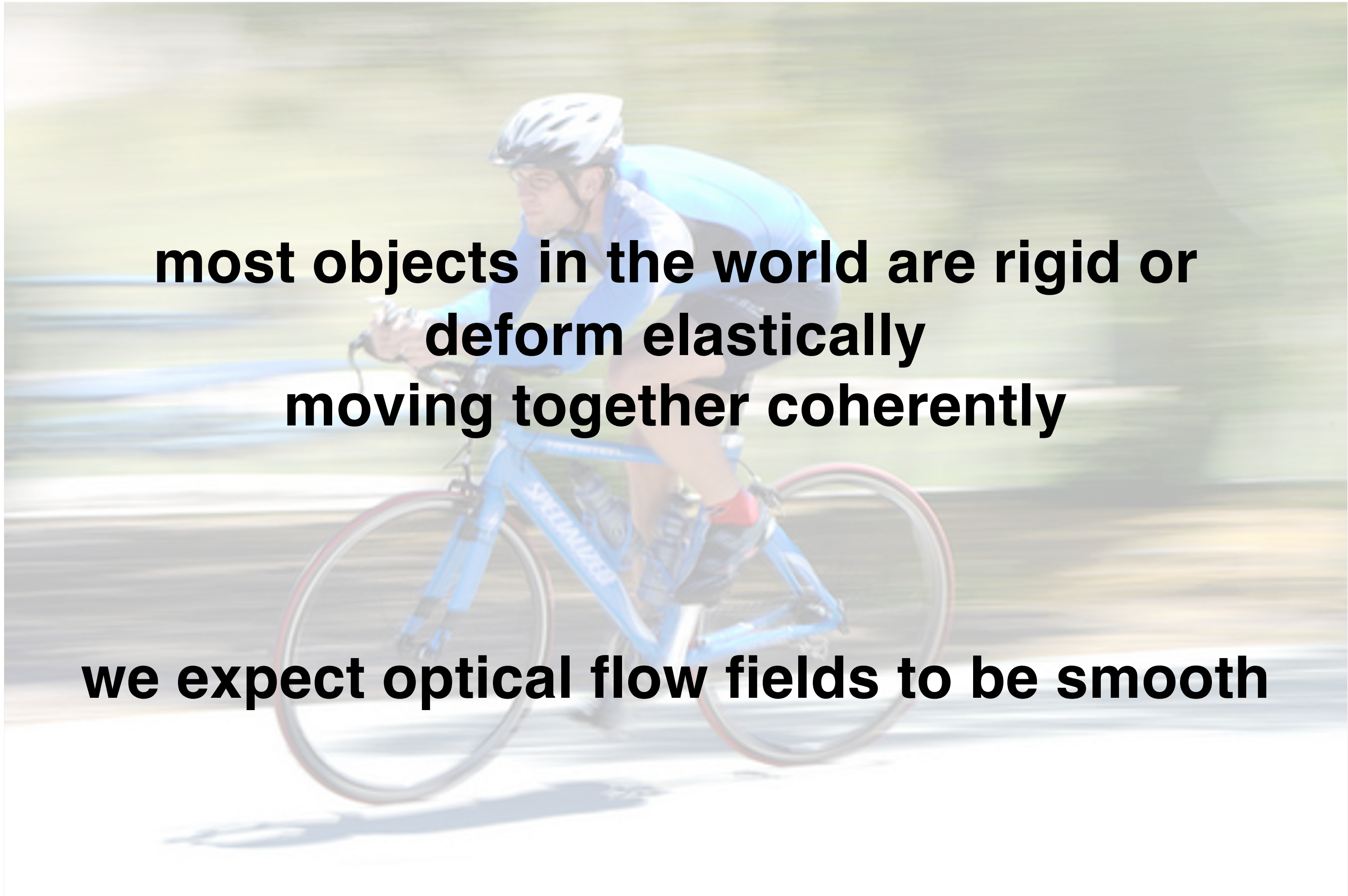
method of differences

'constant' flow

(flow is constant for all pixels)

local method
(sparse)

Smoothness


most objects in the world are rigid or
deform elastically
moving together coherently
we expect optical flow fields to be smooth

Key idea
(of Horn-Schunck optical flow)

Enforce
brightness constancy

Enforce
smooth flow field

to compute optical flow

Key idea
(of Horn-Schunck optical flow)

Enforce
brightness constancy

Enforce
smooth flow field

to compute optical flow

Enforce **brightness constancy**

$$\boxed{I_x u + I_y v + I_t = 0}$$

For every pixel,

$$\min_{u,v} \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

Enforce **brightness constancy**

$$\boxed{I_x u + I_y v + I_t = 0}$$

For every pixel,

$$\min_{u,v} \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

lazy notation for $I_x(i,j)$

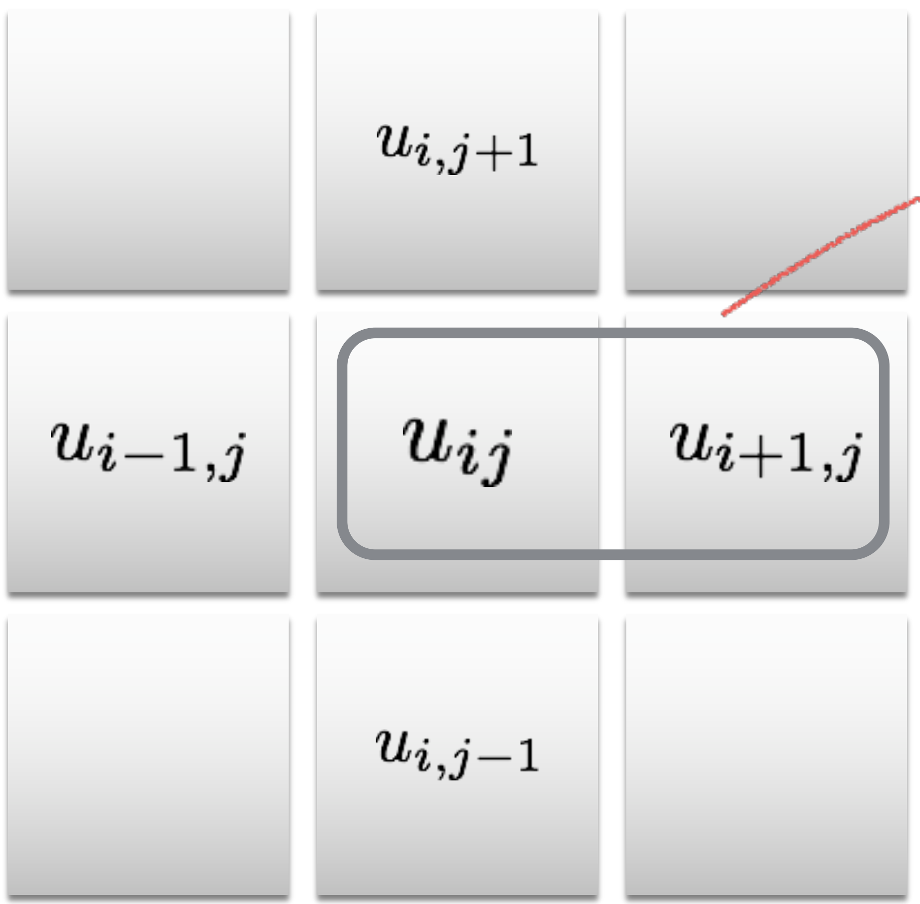
Key idea
(of Horn-Schunck optical flow)

Enforce
brightness constancy

Enforce
smooth flow field

to compute optical flow

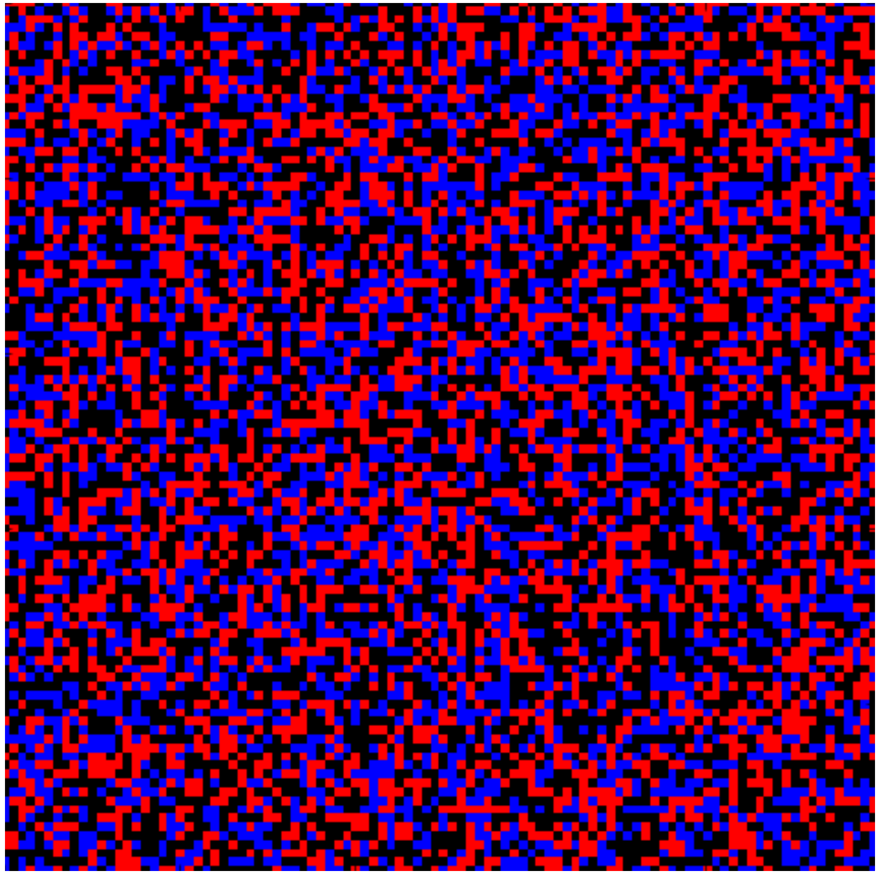
Enforce **smooth flow field**



$$\min_{\boldsymbol{u}} (u_{i,j} - u_{i+1,j})^2$$

u-component of flow

Which flow field optimizes the objective? $\min_{\boldsymbol{u}}(u_{i,j} - u_{i+1,j})^2$



big

small

Key idea
(of Horn-Schunck optical flow)

Enforce
brightness constancy

Enforce
smooth flow field

to compute optical flow

bringing it all together…

Horn-Schunck optical flow

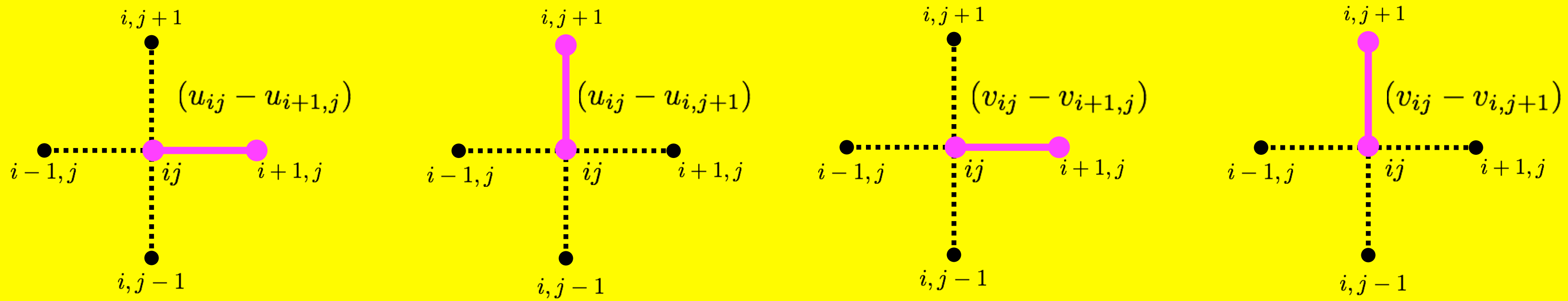
$$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ \overset{\text{smoothness}}{E_s(i,j)} + \underset{\text{weight}}{\lambda} \overset{\text{brightness constancy}}{E_d(i,j)} \right\}$$

HS optical flow objective function

Brightness constancy $$E_d(i,j) = \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

Smoothness

$$E_s(i,j) = \frac{1}{4}\left[(u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right]$$



How do we solve this minimization problem?

$$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

Compute partial derivative, derive update equations

(iterative gradient decent!)

Final Algorithm (after some math)

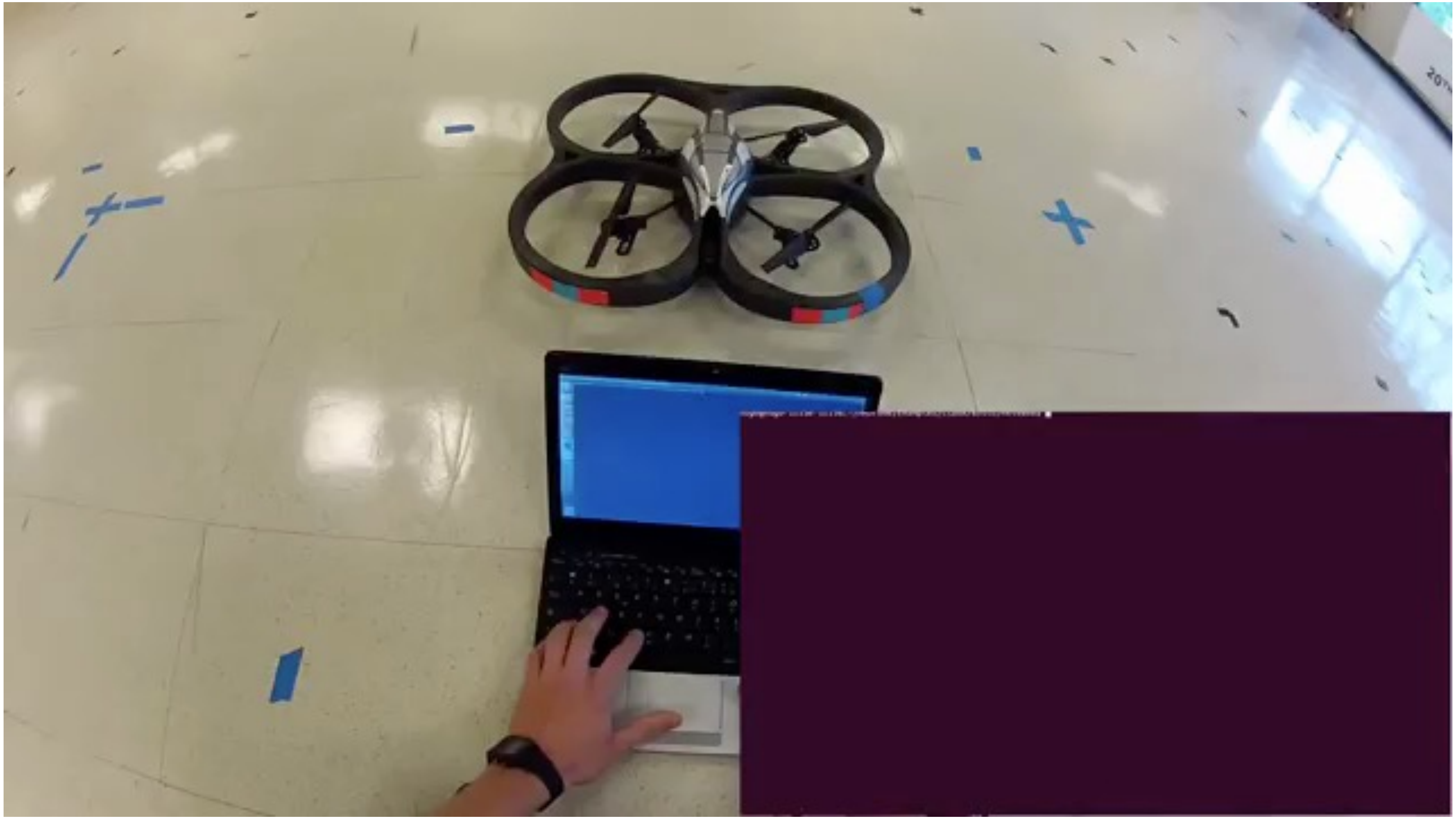
1. Precompute image gradients $I_x \quad I_y$
2. Precompute temporal gradients I_t
3. Initialize flow field $\boldsymbol{u} = \mathbf{0}$
   $\boldsymbol{v} = \mathbf{0}$
4. While not converged

   Compute flow field updates for each pixel:

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x \qquad \hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

Just 8 lines of code!

Optical flow used for feature tracking on a drone



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